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AN EXTENSION OF HÖLDER'S THEOREM ON THE GAMMA FUNCTION

BY

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In memory of Lee A. Rubel

ABSTRACT

The Gamma function is differentially transcendental over certain differential domains of real-valued functions defined on fixed subintervals of $(0, \infty)$.

It is a classical result of Hölder [H] that Euler's Gamma function $\Gamma(z)$ does not satisfy any nontrivial algebraic differential equation with coefficients from $\mathbb{C}(z)$. Since $\mathbb{C}[z]$ is a differential domain—that is, a differential ring that is an integral domain—another way to state this result is that Γ is **differentially transcendental** over $\mathbb{C}[z]$. Extensions of this result to various fields of meromorphic functions other than $\mathbb{C}(z)$ have been established; see in particular Bank and Kaufman [BK]. In this paper, we develop an extension of Hölder's theorem for certain differential domains of real-valued C^∞ functions defined on fixed subintervals of $(0, \infty)$, where the functions in these rings do not necessarily extend meromorphically to regions of the complex plane.

Throughout, let D denote a ring of continuous real-valued functions on $\mathbb{R}$ extending $\mathbb{R}[x]$ (the real polynomial functions) such that for each $f \in D \setminus \{0\}$, the zero set $f^{-1}\{0\}$ is finite and the translation $x \mapsto f(x+1)$ belongs to D. Note that D is an integral domain. (Note also that $\mathbb{R}[x]$ is an example of such a ring; others will be given later.) Given an interval $I \subseteq (0, \infty)$, we let D_I denote the

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integral domain of restrictions $f|I: I \to \mathbb{R}$ for $f \in D$, and we let $D_I\langle\Gamma\rangle$ denote the ring $D_I[\Gamma^{(n)}|I: n \in \mathbb{N}]$ of real-valued functions on I.

Here is the main technical result of this note:

THEOREM: *Fix $I = (c, \infty) \subseteq (0, \infty)$. Assume that $D_I \subseteq C^\infty(I)$ and is closed under taking derivatives, and that $D_I\langle\Gamma\rangle$ is an integral domain. Then $\Gamma|I$ is differentially transcendental over D_I.*

We immediately obtain the following:

COROLLARY: *Fix $I = (c, \infty) \subseteq (0, \infty)$. Assume that $D_I \subseteq C^\infty(I)$ and is closed under taking derivatives, and that $D_I\langle\Gamma\rangle$ is contained in a quasi-analytic class of real functions on I. Then $\Gamma|(a, b)$ is differentially transcendental over $D_{(a,b)}$, for each interval (a, b) with $c \leq a < b \leq +\infty$.*

(A set of functions $S \subset C^\infty(U, \mathbb{R})$ with U an open connected subset of $\mathbb{R}^n$ is a **quasi-analytic class** if for all distinct $f, g \in S$ and each $x \in U$, the Taylor series at x of f and g are distinct; for example, the set of all real analytic functions $f: U \to \mathbb{R}$.) The Corollary thus applies in particular if each $f \in D_I$ is real analytic and D_I is closed under taking derivatives. (See e.g. Chapter 19 of [Ru] for more information about quasi-analytic classes.)

Note: The Corollary shows that certain claims made in the paper [DM1] are false; see [DM2].

We first establish some results about D without the extra assumptions of the Theorem.

LEMMA: *Let $f, g \in D$, $g \neq 0$, and suppose that $f(x+1)/g(x+1) = f(x)/g(x)$ for all x in some nonempty open interval. Then $f = cg$ for some $c \in \mathbb{R}$.*

Proof: The function $x \mapsto g(x)f(x+1) - f(x)g(x+1) \in D$ then vanishes on an interval, so $f(x+1)/g(x+1) = f(x)/g(x)$ for all $x \in \mathbb{R}$ with $g(x), g(x+1) \neq 0$. Now g has finitely many zeros; choose some $N \in \mathbb{N}$ with $g(n) \neq 0$ for all $n \geq N$. Put $c = f(N)/g(N)$. Then $f(n)/g(n) = c$ for all $n \geq N$, and $f - cg \in D$ has infinitely many zeros. Thus, $f = cg$. ∎

LEMMA: *Fix $c \in \mathbb{R} \setminus \{0\}$ and a negative integer k. Then for all $f, g \in D$ with $g \neq 0$, the set of all $x \in \mathbb{R}$ such that $f(x+1)/g(x+1) = f(x)/g(x) + cx^k$ is nowhere dense.*

Proof: Suppose otherwise. Put $u(x) = f(x)/g(x)$ for $x \in \mathbb{R}$ with $g(x) \neq 0$. Arguing similarly as for the previous lemma, we then have $u(x+1) = u(x) + cx^k$ for all nonzero $x \in \mathbb{R}$ with $g(x), g(x+1) \neq 0$. Clearly, u is unbounded at either $x = 0$ or $x = 1$. If the former, then u is unbounded at every nonpositive integer; if the latter, then u is unbounded at every positive integer. But g has only finitely many zeros; contradiction. ∎

PROPOSITION: *Fix* $I = (c, \infty) \subseteq (0, \infty)$. *Then the set* $\{(\Gamma'/\Gamma)^{(n)}|I\colon n \in \mathbb{N}\}$ *is algebraically independent over* D_I.

Using the Lemmas, the proof is an easy modification of part of the proof of Hölder's theorem from Rosenlicht [R, pg. 667], which is just the case that $D = \mathbb{R}[x]$; the details are left to the reader. (One works with the fraction field of D in place of $\mathbb{R}(x)$.)

Proof of Theorem: Fix $I = (c, \infty) \subseteq (0, \infty)$. Assume that D_I is contained in $C^\infty(I)$ and is closed under taking derivatives, and that $D_I\langle\Gamma\rangle$ is an integral domain. Then both D_I and $D_I\langle\Gamma\rangle$ are differential domains; let E and L denote their respective fraction fields. Also, the ring $D_I[(\Gamma'/\Gamma)^{(n)}|I\colon n \in \mathbb{N}]$ is a differential domain; let K denote its fraction field. Since D_I is a differential domain, it suffices to show that the transcendence degree of L over E is infinite, which follows immediately from the Proposition and the fact that $L \supseteq K \supseteq E$. ∎

Here are some examples of rings satisfying the assumptions of the Theorem, that is, subrings of $C^\infty(\mathbb{R})$ containing $\mathbb{R}[x]$ that are closed under taking derivatives and translations $x \mapsto f(x+1)$, and such that the zero set of each nonzero element is finite.

(1) The ring of all real analytic semialgebraic functions on $\mathbb{R}$.
(2) The ring of all real analytic functions f on $\mathbb{R}$ for which there exist sequences of real numbers $\{a_n\}$ and $\{b_n\}$, $N \in \mathbb{N}$ and $r > 0$ (all depending on f) such that $f(x) = \sum a_n x^{N-n}$ for $x < -r$ and $f(x) = \sum b_n x^{N-n}$ for $x > r$.
(3) The ring of all real analytic (respectively, C^∞) functions on $\mathbb{R}$ definable in a given o-minimal (respectively, polynomially bounded o-minimal) expansion of the field of real numbers.

Item (3) above requires perhaps some explanation. For readers familiar with basic model theory, a (first-order) expansion $\mathfrak{R} = (\mathbb{R}, +, \cdot, \dots)$ of the field of real numbers $(\mathbb{R}, +, \cdot)$ is **o-minimal** if every subset of $\mathbb{R}$ (parametrically) definable (in $\mathfrak{R}$) is a finite union of singletons and open intervals. (A brief introduction to o-minimal expansions of $(\mathbb{R}, +, \cdot)$, written for a general mathematical audience, can be found in [DM3]; see also [D], [DM1] for further information and examples.) In particular, if $f\colon \mathbb{R} \to \mathbb{R}$ is analytic, definable in an o-minimal expansion of $(\mathbb{R}, +, \cdot)$ and $f^{-1}\{0\}$ is infinite, then $f^{-1}\{0\}$ contains an open interval; hence $f = 0$. An expansion $\mathfrak{R}$ of $(\mathbb{R}, +, \cdot)$ is **polynomially bounded** if for every definable function $f\colon \mathbb{R} \to \mathbb{R}$ there exists $n \in \mathbb{N}$ (depending on f) such that $f(x) = O(x^n)$ as $x \to +\infty$. The ring of all C^∞ functions on $\mathbb{R}$ definable in a given polynomially bounded o-minimal expansion of $(\mathbb{R}, +, \cdot)$ is a quasi-analytic class, so again nonzero functions from such a ring have finitely many zeros. (See [M1] and [M2] for more information on polynomially bounded o-minimal structures.) Example (1) is a special case of (3), with $\mathfrak{R} = (\mathbb{R}, +, \cdot)$. Example (2) is a subring of a special case of (3); see [D].

Let us conclude with some remarks on the extent to which the Theorem is optimal.

We can weaken the assumption on the zero sets: An examination of the proofs of the Lemmas (and the Proposition) shows that we need only assume that the zero set of each $f \in D \setminus \{0\}$ be nowhere dense and contain only finitely many *integer* zeros. However, some such condition on the integer zeros is certainly necessary for the Theorem to hold. Recall that $1/\Gamma$ is real-valued and real analytic on $\mathbb{R}$, with a zero at every nonpositive integer. Consider the differential domain $R = \mathbb{R}[x]\left[(1/\Gamma)^{(n)}(x+m)\colon m, n \in \mathbb{N}\right]$ of real analytic functions on $\mathbb{R}$. Note that if $f \in R$, then $x \mapsto f(x+1)$ belongs to R, and $f^{-1}\{0\}$ consists of isolated points unless $f = 0$. Now $1/\Gamma$ belongs to R, so $\Gamma|(0,\infty)$ is even linear over the ring $\{f|(0,\infty)\colon f \in R\}$.

What about a relaxation of the assumption that the functions in D be defined on all of $\mathbb{R}$? The most natural idea would be to consider differential rings of real analytic functions on $[0,\infty)$ that satisfy all other assumptions on the ring D. (We say that a function $f\colon X \to \mathbb{R}$ with $X \subseteq \mathbb{R}$ is real analytic if there is an open neighborhood U of X and a real analytic function $F\colon U \to \mathbb{R}$ with $f = F|X$.) However, this idea fails: Let R be as in the preceding paragraph and consider the differential ring $S = \{f|[0,\infty)\colon f \in R\}$ of real analytic functions on $[0,\infty)$. The germs at $+\infty$ of the functions in S lie in a Hardy field; see [R]. Thus, for each $f \in S \setminus \{0\}$ there is a positive real number a (depending on f) such that

$f^{-1}\{0\} \subseteq [0, a]$. Then $f^{-1}\{0\}$ must be finite, since f is real analytic on $[0, \infty)$ and $f \neq 0$. As before, $\Gamma|(0, \infty)$ is linear over $\{f|(0, \infty): f \in R\}$. A similar argument shows that the conclusion of the Theorem also fails in general for differential rings of real analytic functions defined on any fixed *proper* subinterval (a, ∞) of $\mathbb{R}$, with all other assumptions on D satisfied.

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